A Fuzzy Interpolation Algorithm Closed over CNF Sets

Domonkos Tikk, Tamás D. Gedeon and Péter Baranyi

Abstract

Fuzzy rule interpolation is a reasoning technique that is designed to cope with sparse rule bases. The existing methods suffer from either one of the following deficiencies: problem of abnormal conclusion and high computational cost. This paper introduces a new interpolation algorithm that eliminates these shortcomings. It is proven that the method is closed over convex and normal fuzzy (CNF) sets. Characteristics of the proposed method are also investigated.

Keywords: Fuzzy Control, Fuzzy Rule Interpolation, Problem of Abnormality, Computational Efficiency.

1. Introduction

Fuzzy rule interpolation is a technique invented for handling sparse rule bases, where classical inference methods of fuzzy control (Zadeh, Mamdani and Sugeno) are not suitable.

The first published approach [2] proposed by Kóczy and Hirota can result in abnormal fuzzy set under certain configuration of the inputs. Although several different methods were introduced so far inspired also by this defect [1,4], none of them could maintain the advantageous complexity property of the KH approach still recent past. In [9] a modification of the KH fuzzy rule interpolation method was presented for eliminating the problem of abnormality. For brevity, we will refer to this as MACI (Modified Alpha Cut based Interpolation) method. The purpose of this paper is to characterize and analyze the MACI method.

2. Fuzzy Rule Interpolation

The introduction of fuzzy rule interpolation methods was originally motivated by the will of reducing the number of the rules and, subsequently, the complexity of the resulting fuzzy system by means of the omission of

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redundant rules, which may result in sparse rule base [2]

KH method was the first published fuzzy rule interpolation. This technique is based on the $\alpha - cut$ distances of convex and normal fuzzy sets. The method generates the fuzzy conclusion by means of its $\alpha - cuts$ based on the Extension and the Resolution Principles.

The simplest approach based on this idea is the linear KH interpolation, which takes into account two (closest) rules flanking the actual observation and calculates the conclusion between the corresponding rule consequents as

$$B_{\alpha,C}^{*} = \frac{\dfrac{B_{1\alpha,C}}{d_C(A_{\alpha}^{*},A_{1\alpha})} + \dfrac{B_{2\alpha,C}}{d_C(A_{\alpha}^{*},A_{2\alpha})}}{\dfrac{1}{d_C(A_{\alpha}^{*},A_{1\alpha})} + \dfrac{2}{d_C(A_{\alpha}^{*},A_{2\alpha})}}, \quad C \in \{L,U\} \qquad (1)$$

where A_i and B_i ($i=1,2$) denote antecedents neighboring the observation A^* and corresponding consequents, respectively, subscripts L and U in C refer to the minimum and maximum of respective $\alpha - cuts$, and $d_C : \Re \times \Re \to (0,+\infty)$, $C \in \{L,U\}$ is an appropriate distance function (cf. [3]). This should be calculated for every important cut, e.g. in the case of triangular and trapezoidal shaped membership functions $\alpha = 0$ and 1 (see [5]).

The linear KH method, although being computationally efficient for practical piecewise linear shaped fuzzy sets, has the shortcoming that it can produce abnormal conclusion (Figure 1).
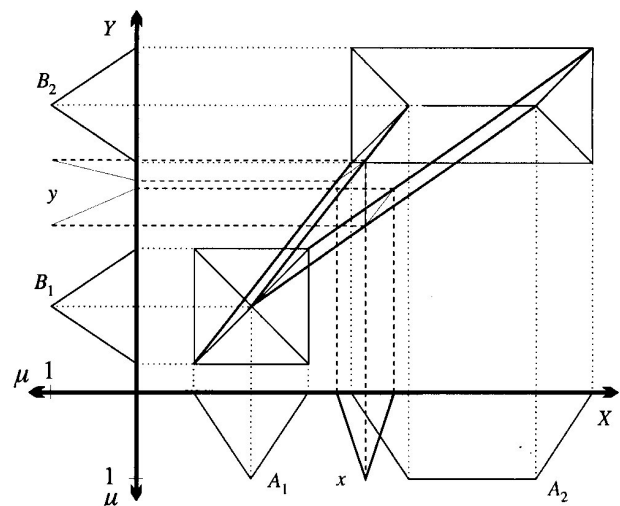

Figure 1. Abnormal conclusion produced by the KH method

In order to alleviate this problem condition was imposed by e.g. Kóczy and Kovács [5], or Shi and Mizumoto [6] so as to yield a normal fuzzy set. The application of these condition leads to restriction of the shape of the rules and of the observation that might be an obstacle of practical applications in some contexts. Other authors proposed conceptually different methods, but these have the drawbacks of immense computational need.

3. The MACI Method

The MACI method to be presented modifies the KH fuzzy rule interpolation method in that manner, that the new fuzzy rule interpolation method should always give fuzzy set as conclusion.

We recall that the KH fuzzy rule interpolation method determines the conclusion by its $\alpha - cuts$. The conditions concerning the conclusion ensuring its normality and convexity can be characterized as follows in terms of the $\alpha - cuts$. It should be satisfied for every $\alpha_1 \langle \alpha_2 \in [0,1]$ that the support of $\alpha - cut$ should contains the support of $\alpha_2 - cut$ (convexity). The normality is guaranteed by the premise of the KH fuzzy rule interpolation method, namely, that it is suitable for CNF sets. The above condition can be expressed as

$$\min B_{\alpha1}^* \leq \min B_{\alpha2}^* \leq \max B_{\alpha2}^* \leq \max B_{\alpha1}^* \qquad (2)$$

for all $\alpha_1 \langle \alpha_2 \in [0,1]$.

We use vector description of fuzzy sets (Yam 1997; [10]) to solve the abnormality problem. A piecewise linear fuzzy set is described by vector of its characteristic points. We can split a convex fuzzy set into two parts: the left and the right flank. They are connected at the reference point a_0, which is the most important point of the fuzzy set with the highest membership degree. The reference point is usually the center of the core (see also Figure 2). The points of the left (right) flank and are index by descending (ascending) integer from 0.
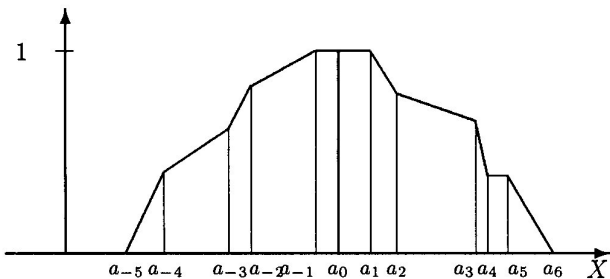

Figure 2. Abnormal conclusion produced by the KH method

To capture the main idea of the MACI method, first, the simplest case is presented, where isosceles triangular shaped fuzzy sets and single dimensional input space are

considered.

A. The Skeleton of the MACI Method

Triangular fuzzy set A can be described with vector representation as $A = \{a_{-1}, a_0, a_1\}$, a_{-1}, a_0, a_1 should satisfy to ensure the normality of A. Henceforth, without the loss of generality, we present the algorithm for the right flank (for the analogous left flank, see [7]), which is represented by the vector

$$\mathbf{a} = \begin{bmatrix} a_0 & a_1 \end{bmatrix}^T \qquad (3)$$

Let us suppose further, that two fuzzy rules are given as $A_1 \rightarrow B_1$ and $A_2 \rightarrow B_2$ with the observation located in between A_1 and A_2.

The conclusion according to the original $\alpha - cut$ based KH fuzzy rule interpolation method by means of the vector representation form can be expressed (for the right flank) as:

$$\mathbf{b}^* = (\mathbf{I}_2 - \mathbf{I}_2\mathbf{\Lambda})\mathbf{b}_1 + \mathbf{I}_2\mathbf{\Lambda}\mathbf{b}_2 \qquad (4)$$

where $\mathbf{I}_2$ is the 2 by 2 identity matrix and

$$\mathbf{\Lambda} = [\lambda_0 \quad \lambda_1], \quad \lambda_k = \frac{a_k^* - a_{1k}}{a_{2k} - a_{1k}}, k = 0,1. \qquad (5)$$
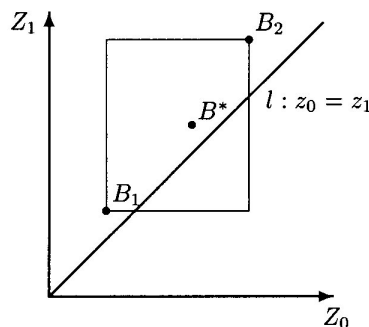

Figure 3. The representation of the right slope of consequents B_1, B_2, and the conclusion B^* (satisfying the CNF condition) as the values of their characteristic points.

Figure 3 depicts vectors $\mathbf{b}_1, \mathbf{b}_2$, and $\mathbf{b}^*$ (representing the right flank of fuzzy sets B_1, B_2, and B^*) in the two dimensional space $Z_0 \times Z_1$, where the coordinates are the values of their vector representation. Because of the premises of the original KH fuzzy rule interpolation method $a_{1k} \langle a_k^* \langle a_{2k}$, the fractions $\lambda_k (k = 0,1)$ are nonnegative numbers in unit the interval. Hence the location of B^* must be in the rectangle drawn by the thin lines in figure 3. In order to fulfill (2) the coordinates of the conclusion B^* should satisfy $b_0^* \leq b_1^*$. That is, it should be above the straight line $l : z_0 = z_1$. If the rectangular is crossed by the line l than

there is always a chance for abnormal conclusion. The whole rectangle is above the line *l* if the sets B_1 and B_2 overlap.

The essential idea of the MACI method is the transformation of the points B_1, B_2 in another coordinate system where the abnormality can be excluded. In this case axis Z_0 is substituted by the straight $l : z_0 = z_1$ and axis Z_1 remains unchanged. Coordinates of B_1 and B_2 are nonnegative in the new coordinate system. The coordinates of the conclusion are computed in the transformed coordinate system that ensures that it will never be under the straight *l*. Then we transform back the resulting conclusion to the original coordinate system. The fulfillment of inequality $b_0^* \le b_1^*$. is ensured by restricting the search for the conclusion to the area bounded by the straight lines Z_1 and *l*.

Because the lack of space we briefly resume the steps of the transformation. For further details on the coordinate transformation we refer to [7].
The transformation matrix describes the above change of axis can be defined as

$$T = \begin{bmatrix} \sqrt{2} & 0 \\ -1 & 1 \end{bmatrix} \qquad (6)$$

The transformed conclusion is calculated based on (4) (superscript prime denotes the vectors in the transformed space).

$$\mathbf{b}^{*\prime} = (\mathbf{I} - \mathbf{I}\Lambda)\mathbf{b}_1' + \mathbf{I}\Lambda\mathbf{b}_2' \qquad (7)$$

The premises ensure that the new coordinates are nonnegative numbers. Finally, using the inverse of matrix *T*, we can transform back the conclusion.

Analogous method can be applied for the left flank (see [7]). Note that due to the construction, the left and the right flank are connected, and the coordinates of core do not differ from the one by the original KH method.

A.1 The Generalizations of the MACI Method

The above described algorithm can be carried over for arbitrary piecewise linear CNF sets. Naturally, with the increasing number of characteristic points, the coordinate transformation (i.e. the transformation matrix **T**}) becomes more complicated. As a consequence, the computational cost also increases proportionally.

The coordinate axes are now $Z_0, \ldots, Z_n$, where *n* is the number of the characteristic points in the right flank. (Here we only present the algorithm for the right flank). The conclusion should fulfill the condition (2) changes to $b_i^* \le b_j^* \forall i \le j \in [0,n]$. This can be ensured by changing the axis Z_i for the straight Z_i' defined as

$$Z_i' = \{(z_i, \ldots, z_n) \mid z_l = z_m, l, m \in [i,n]\}. \qquad (8)$$

Hence Z_0' is the straight line satisfying $z_i = z_j$ $i, j \in [0,n]$, while Z_n' remains unchanged.

The transformation matrix is then

$$T = \begin{bmatrix} \sqrt{n+1} & 0 & 0 & 0 & \cdots & 0 \\ -\sqrt{n} & \sqrt{n} & 0 & 0 & \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & 0 & -\sqrt{2} & \sqrt{2} & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 1 \end{bmatrix}. \qquad (9)$$

For more details see [7].

Further, using the same procedure as in the two dimensional case, we obtain the transformed conclusion as

$$\mathbf{b}^{*'}_k = (1 - \lambda_k)\mathbf{b}_{1k}' + \lambda_k \mathbf{b}_{2k}', \qquad (10)$$
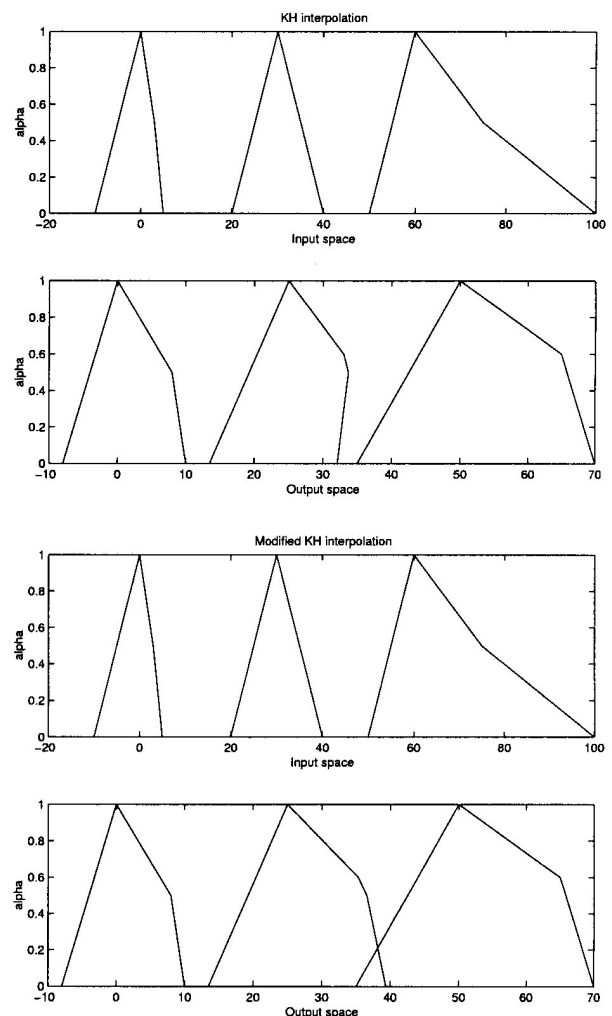

Figure 4. A case where MACI resolves the abnormality problem generated by KH method

where λ_k are as in (5) with range $k \in [0,n]$. Finally, by means of the inverse matrix of (9) we end up with the final conclusion.

Summarizing the above the following theorem can be stated:

Theorem 3.1 [7] The MACI method described so far always gives CNF set as conclusion.

In Figure 4 depicts a situation where the MACI method corrects the abnormal conclusion of the KH method.

In the case of piecewise linear membership function the breakpoint level of the fuzzy set in the input space can be different from that of one the output space. In (10) the coefficients λ_k correspond to the kth coordinate of the antecedents and the observation, but the calculated value corresponds to the kth coordinate of the conclusion. So, for clarity, a common breakpoint level set should be determined for both spaces, which is the union of (perhaps different) breakpoint level fuzzy sets for each variable. As an example see Figure 5.
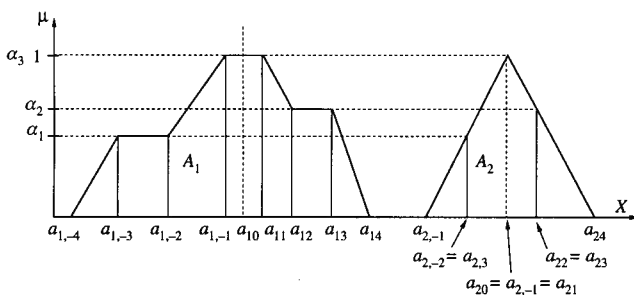

Figure 5. Determination of characteristic points

By increasing the nodes in the distribution of range [0,1] one can get finer result for the required conclusion. For arbitrary continuous (convex and normal) membership function the value of the conclusion can be approximated by increasing $n \rightarrow \infty$.

Multi-variable antecedent case can be handled analogously as the transformation described so far affects only the consequent part. Common combined antecedent fuzzy sets (and observation) can be calculated from the corresponding antecedents (observation) of each variable using Minkowski-type distance, where the weights are identically one (w=1), in order to preserve the linearity of the fuzzy rule interpolation method. Hence, for example, the coordinates of the common combined observation can be calculated as $a_k^* = \frac{1}{r}\sum_{i=1}^{r} a_{ik}^*$, where r is the number of variables, and a_{ij}^* is the jth coordinate of the observation of the ith variable.

The MACI algorithm can further be generalized by taking more than two rules into account for the determination of the conclusion. Farther an antecedent of a rule is from the observation less significant its role plays the corresponding consequent in the determination of the conclusion. Hence, each coordinate values are weighted with the reciprocal of the fuzzy distance of the appropriate characteristic points. It can be shown that this general approach is behaves as universal approximator in the set of continuous function with respect to the supremum norm. This issue was addressed in [8].

4. Conclusions

In this paper a new technique for fuzzy rule interpolation has been presented. It is originated from the first published α-cut based fuzzy rule interpolation, and it was proposed to combine the advantageous complexity behavior of the original KH approach and at the same time alleviate the disadvantage that for certain configuration it produces abnormal conclusion.

Acknowledgements

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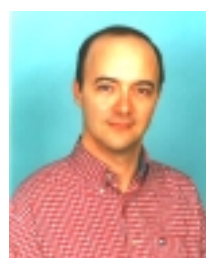
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